

# Weyl functions, inverse problem and special solutions for the system auxiliary to the nonlinear optics equation

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## Abstract

A Borg-Marchenko type uniqueness theorem (in terms of the Weyl function) is obtained here for the system auxiliary to the  $N$ -wave equation. A procedure to solve inverse problem is used for this purpose. The asymptotic condition on the Weyl function, under which the inverse problem is uniquely solvable, is completed by the new and simple sufficient condition on the potential, granting the fulfillment of this asymptotic condition. The evolution of the Weyl function is discussed and the solution of an initial-boundary value problem for the  $N$ -wave equation follows. Explicit solutions of the system are obtained. System with a shifted argument is treated.

## 1 Introduction

The well-known integrable nonlinear optics ( $N$ -wave) equation has the form

$$[D, u_t] - [\check{D}, u_x] = \left[ [D, u], [\check{D}, u] \right], \quad [D, u_t] := Du_t - u_tD, \quad (1.1)$$

where  $D = D^*$  and  $\check{D} = \check{D}^*$  are  $m \times m$  diagonal matrices,  $u_x$  and  $u_t$  are the partial derivatives of the  $m \times m$  matrix function  $u(x, t) = u(x, t)^*$ . Equation (1.1) is the compatibility condition of the auxiliary systems

$$Y_x = G(x, t, z)Y, \quad Y_t = \check{G}(x, t, z)Y,$$

where

$$G := izD - \zeta(x, t), \quad \check{G} := iz\check{D} - \check{\zeta}(x, t), \quad \zeta = [D, u], \quad \check{\zeta} = [\check{D}, u]. \quad (1.2)$$

The case of three waves ( $n = 3$ ) interaction have been treated in the seminal paper [44], using zero curvature representation. It was one of the first models to demonstrate the advantages of the zero curvature representation. (See also [3] for the  $N$ -wave case.) Several years later model (1.1) proved to be the first integrable system, which was studied via Riemann-Hilbert problem approach [41]. The  $N$ -wave equation is actively studied and used in optics, fluid dynamics and plasma physics. It describes a variety of phenomena, including patterns and various instabilities, and is also closely related to many other important nonlinear integrable wave equations. For these and other connections and applications see, for instance, the books and recent publications [1, 2, 4, 5, 13, 14, 16, 17, 26] and references therein.

We shall consider the auxiliary to (1.1) system on the semiaxis:

$$Y_x(x, z) = (izD - \zeta(x))Y(x, z), \quad x \geq 0 \quad \left( Y_x := \frac{dY}{dx} \right), \quad (1.3)$$

where, without loss of generality, we assume  $D > 0$ :

$$D = \text{diag} \{d_1, d_2, \dots, d_m\}, \quad d_1 > d_2 > \dots > d_m > 0; \quad \zeta(x) = -\zeta^*(x). \quad (1.4)$$

Here  $\text{diag}$  stands for diagonal matrix, and  $\zeta(x)$  is an  $m \times m$  matrix function. The study of system (1.3) is basic for study and construction of the solutions of the  $N$ -wave equation (for study of the difficult and important initial-boundary value problem for the  $N$ -wave equation, in particular). Moreover, this system is a natural generalization of the well-known Dirac (called also Zakharov-Shabat or AKNS) system. Various useful results and references on the scattering problem for system (1.3) one can find in [2, 6, 7, 8, 9, 18, 27].

An extensive amount of research on the scattering problem for system (1.3) with the complex-valued entries  $d_k$  of  $D$  was done by R. Beals and R.R. Coifman. The unique solvability of the problem on a suitable dense set of scattering data was obtained, in particular. The case (1.4) of the positive  $D$ , that is considered here, is less general, and therefore more explicit description of the class of data for which the solution of the inverse problem exists and a close to the classical procedure for solving this inverse problem have been obtained in [30, 31, 33] in terms of the generalized Weyl functions. Notice that various generalizations of the Weyl functions are successfully used both in the inverse scattering and in the inverse spectral problems.

Here we develop further the results from [30, 31, 33]. It is shown that the asymptotic conditions (2.9) and (2.10) on the generalized Weyl function, under which the inverse problem is uniquely solvable, are automatically fulfilled for the integrable and two times differentiable  $\zeta$  with integrable derivatives. If conditions (2.9) and (2.10) are valid for the Weyl function of the initial system, they are valid also for systems with a shifted argument, and they hold under the evolution of the Weyl function too. In this way we obtain sufficient conditions, when the inverse problem for system (1.3) and an initial-boundary value problem in the quarterplane for the  $N$ -wave equation (1.1) are solvable, and our procedure for solving these problems works.

A recent series of papers by F. Gesztesy, B. Simon and coauthors on the high energy asymptotics of the Weyl functions and local Borg-Marchenko type uniqueness results has initiated a growing interest in this important domain (see [12, 20, 22, 35, 42, 43] and references therein). The Weyl-Titchmarsh theory for a non-self-adjoint case (the skew-self-adjoint Dirac type system) has been studied in [11, 23, 29] and the Borg-Marchenko type results for this system have been published in [36]. In this paper we obtain a Borg-Marchenko type theorem for another important non-self-adjoint case, that is, system (1.3).

Construction of the explicit solutions of the inverse problems and non-linear equations is of great interest and Bäcklund-Darboux transformation is one of the most fruitful methods to do it. The initial approach by Bäcklund and Darboux have been greatly generalized and developed (see, for instance, [10, 15, 19, 21, 25, 45] and references therein). We consider some applications

and developments of the version of the Bäcklund-Darboux transformation, which is called GBDT (see [32, 34, 37]).

Section 1 is an introduction and Section 2 contains preliminaries to make the paper self-sufficient. Section 3 contains Theorem 3.1, which states that for the two times differentiable  $\zeta$  with integrable derivatives asymptotic conditions (2.9) and (2.10) on the Weyl functions are fulfilled. It contains also a Borg-Marchenko type Theorem 3.2. Weyl functions for systems with a shifted argument are treated in Section 4. Evolution of the Weyl function and solution of the initial-boundary value problem are described in Remark 4.9. GBDT for system (1.3) is discussed in Section 5. Lemma 4.7 is proved in Appendix A and formula (5.6) is proved in Appendix B.

The space  $L^1(a, b) = L^1_{m \times m}(a, b)$  of  $m \times m$  matrix functions on  $(a, b)$  is equipped with the norm  $\|f\|_1 = \int_a^b \|f(x)\| dx$ , where the matrix norm is defined in terms of the trace  $\text{Tr}$  by the equality  $\|f(x)\| = \left( \text{Tr}(f(x)^* f(x)) \right)^{\frac{1}{2}}$ . We denote by  $C^k(a, b)$  the class of  $k$  times differentiable matrix functions.

## 2 Preliminaries

We shall consider system (1.3), such that the inequalities

$$\sup_{0 < x < l} \|\zeta(x)\| < \infty \quad (2.1)$$

are true for each  $l < \infty$ . The  $m \times m$  fundamental solution  $w$  of system (1.3) is normalized by the condition

$$w(0, z) = I_m, \quad (2.2)$$

where  $I_m$  is the  $m \times m$  identity matrix. A generalized Weyl function, later called a Weyl function, is introduced for (1.3) slightly different from [33].

**Definition 2.1** *A Weyl function of system (1.3) is an  $m \times m$  matrix function  $\varphi$ , such that for some  $M > 0$  it is analytic in a lower semiplane  $\Im z < -M$ , and the inequalities*

$$\sup_{x \leq l, \Im z < -M} \|w(x, z)\varphi(z) \exp(-ixzD)\| < \infty \quad (2.3)$$

*hold for all  $l < \infty$ .*

System (1.3) with a bounded on the semiaxis potential  $\zeta$ :

$$\sup_{0 < x < \infty} \|\zeta(x)\| \leq M_0, \quad (2.4)$$

was treated in Lemma 1.1 [33]. For this case, it was proved that a Weyl function always exists and admits normalization

$$\varphi_{kj}(z) \equiv 1 \text{ for } k = j, \quad \varphi_{kj}(z) \equiv 0 \text{ for } k > j. \quad (2.5)$$

Moreover, by Theorems 1.1 and 2.1 [33] a normalized, as in (2.5), Weyl function of system (1.3) with a bounded on the semiaxis potential is unique. This Weyl function satisfies for some  $r > 0$  the inequality

$$\int_0^\infty \left( \exp(ix\bar{z}D) \right) \varphi(z)^* w(x, z)^* w(x, z) \varphi(z) \exp \left( x(-izD - rI_m) \right) dx < \infty, \quad (2.6)$$

where  $\Im z < -M$ . Inequality (2.6) is somewhat similar to the inequalities characteristic for the classical Weyl functions.

**Definition 2.2** *The inverse spectral problem (ISpP) for system (1.3), which satisfies (2.1), is the problem to recover the system, i.e., to recover the matrix function  $\zeta$ , such that*

$$\zeta(x) = -\zeta(x)^*, \quad \zeta_{kk}(x) = 0, \quad (2.7)$$

*from a Weyl function. We shall denote by  $\Omega$  the operator mapping the pair  $D$  and  $\varphi(z)$  into  $\zeta$ , that is,  $\Omega(D, \varphi) = \zeta$ .*

We no longer assume that  $\varphi(z)$  satisfies (2.5).

**Theorem 2.3** [33] *For any matrix function  $\varphi(z)$ , which is analytic and bounded in the semiplane  $\Im z < -M$ , and which has the property*

$$\int_{-\infty}^\infty (\varphi(z) - I_m)^* (\varphi(z) - I_m) d\lambda < \infty \quad (z = \lambda - i\eta, \quad \lambda \in \mathbb{R}, \quad \eta > M), \quad (2.8)$$

*there is at most one solution of the ISpP, i.e.,  $\Omega(D, \varphi)$  is unique.*

The existence of the ISpP solutions was proved in Theorem 1.3 [33] under stricter conditions. Namely, we require that

$$\sup \left\| z \left( \varphi(z) - I_m \right) \right\| < \infty \quad (\Im z < -M), \quad (2.9)$$

and that for some matrix  $\alpha$  for all lines  $z = \lambda - i\eta$  with fixed values  $\eta > M$ , we have

$$z \left( \varphi(z) - I_m - \alpha/z \right) \in L_{m \times m}^2(-\infty, \infty). \quad (2.10)$$

Without loss of generality we suppose also that

$$\det \varphi(z) \neq 0. \quad (2.11)$$

**Theorem 2.4** [33] *Let the analytic matrix function  $\varphi$  satisfy (2.9)-(2.11). Then  $\varphi$  is a Weyl function of a unique system (1.3), such that (2.7) holds.*

Finally, let us give the procedure to recover  $\zeta$  from  $\varphi$  (see the proof of Theorem 1.3 [33]). First notice that relations (2.9)-(2.11) yield

$$\sup \left\| z \left( \varphi(z)^{-1} - I_m \right) \right\| < \infty \quad (\Im z < -M), \quad (2.12)$$

and

$$z \left( \varphi(z)^{-1} - I_m + \alpha/z \right) \in L_{m \times m}^2(-\infty, \infty). \quad (2.13)$$

Therefore, we can introduce the  $m \times m$  matrix function

$$\Pi(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{z} \left( \exp(izx D) \right) \varphi(z)^{-1} d\lambda \quad (z = \lambda - i\eta, \quad \eta > M, \quad x \geq 0), \quad (2.14)$$

where the integral is understood as the matrix function, which entries are the norm limits in  $L^2(0, l)$  of the integrals from  $a$  to  $b$  ( $a \rightarrow -\infty$ ,  $b \rightarrow \infty$ ) of the entries of  $z^{-1} \left( \exp(izx D) \right) \varphi(z)^{-1}$ . So  $\Pi(x) \in L_{m \times m}^2(0, l)$  is defined on each interval  $(0, l)$ . Recall that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp(izx D)/z d\lambda \equiv I_m \quad (x \geq 0). \quad (2.15)$$

According to (2.12)–(2.15),  $\Pi(x)$  is twice differentiable and the following properties hold:

$$\Pi(0) = I_m, \quad \Pi'(0) = -iD\alpha, \quad e^{-xMD} \Pi'(x) \in L_{m \times m}^2(0, \infty) \quad (\Pi' = \Pi_x), \quad (2.16)$$

$$e^{-xMD} \Pi''(x) \in L_{m \times m}^2(0, \infty). \quad (2.17)$$

Now, let us introduce the linear operator  $S_l$ , which is bounded on  $L_m^2(0, l)$ :

$$S_l f = D^{-1}f + \int_0^l s(x, u) f(u) du, \quad (2.18)$$

where  $s(x, u) = \{s_{kj}(x, u)\}_{k,j=1}^m$ ;  $0 \leq x, u \leq l$ ;

$$s_{kj}(x, u) = \int_{\gamma} \theta_{kj}(v, u + d_k d_j^{-1}(v - x)) dv + \begin{cases} d_k^{-1} \Pi'_{kj}(x - d_j d_k^{-1} u) & \text{for } u \leq d_k d_j^{-1} x, \\ d_j^{-1} \bar{\Pi}'_{jk}(u - d_k d_j^{-1} x) & \text{for } d_k d_j^{-1} x < u; \end{cases} \quad (2.19)$$

$$\theta(x, u) = \{\theta_{kj}(x, u)\}_{k,j=1}^m = \Pi'(x) [\Pi'(u)]^* D^{-1},$$

$\gamma$  is the interval  $[\max(0, x - d_j d_k^{-1} u), x]$ . Sometimes we omit "l" in  $S_l$  and write just  $S$ . The operator  $S$  satisfies the operator identity

$$AS - SA^* = i \Pi \Pi^*, \quad (2.20)$$

where the operator  $A$  acts in  $L_m^2(0, l)$ :  $Af = iD \int_0^x f(u) du$ , and  $\Pi$  acts from  $\mathbb{C}^m$  into  $L_m^2(0, l)$ :  $\Pi g = \Pi(x)g$ . Using the operator identity, it was shown in [33] that

$$S_l \geq \varepsilon(l)I \quad (\varepsilon > 0, \quad I - \text{identity operator}). \quad (2.21)$$

By (2.16)–(2.21) one gets the representation [24]

$$S^{-1} = V^*V, \quad Vf = D^{1/2}f + \int_0^x V(x, u) f(u) du, \quad (2.22)$$

where  $\int_0^l \int_0^x V(x, u)^* V(x, u) du dx < \infty$ . The operator  $V^{-1}$  admits representation

$$V^{-1}f = D^{-1/2}f + \int_0^x \Gamma(x, u)f(u)du. \quad (2.23)$$

Moreover, according to formula (1.60) in [33] we get

$$\begin{aligned} V(u, x)^* &= -S_u^{-1}s(x, u)D^{1/2} & (0 < x \leq u), \\ \Gamma(x, u) &= s(x, u)D^{1/2} + \int_0^u s(x, v)V(u, v)^* dv & (x \geq u > 0), \end{aligned} \quad (2.24)$$

where  $S_u^{-1}$  is applied to  $s(x, u)$  columnwise. Formula (2.24) implies

$$\Gamma(l, l) = D^{-1}(S_l^{-1}s(x, l))(l) D^{1/2}. \quad (2.25)$$

The potential  $\zeta$  can be easily recovered from  $\Gamma(l, l)$  [33]:

$$\zeta(l) = (\Gamma(l, l) - D\Gamma(l, l)D^{-1})D^{1/2}. \quad (2.26)$$

Thus, formulas (2.14), (2.18), (2.19), (2.25), and (2.26) determine  $\zeta$ . The proof of (2.26) in [33] is based on the representation of the fundamental solution  $w$ :

$$w(x, z) = D^{-\frac{1}{2}}\beta(x)w_A(x, z), \quad (2.27)$$

where  $w_A$  is the transfer matrix function in the Lev Sakhnovich form [38]-[40]:

$$w_A(l, z) = I_m + iz\Pi^*S^{-1}(I - zA)^{-1}\Pi, \quad (2.28)$$

and  $\beta(x) = (V\Pi)(x)$ . By the last relation, in view of (2.22) and (2.24), we have

$$\beta(l) = D^{\frac{1}{2}}\left(\Pi(l) - (\Pi(u), S^{-1}s(u, l))\right), \quad (2.29)$$

where  $(\cdot, \cdot)$  denotes a matrix with the entries, which are scalar products of the columns of matrices in the parenthesis, that is

$$(\Pi(u), S^{-1}s(u, l)) = \int_0^l (S^{-1}s(u, l))^* \Pi(u) du. \quad (2.30)$$

Now, consider system (1.3) on the whole axis, and let  $m \times m$  matrix function  $W$  satisfy (1.3). Then, matrix function  $\mathcal{M}(x, z) = W(x, z) \exp(-ixzD)$  satisfies equation

$$\mathcal{M}_x(x, z) = iz \left[ D, \mathcal{M}(x, z) \right] - \zeta(x) \mathcal{M}(x, z), \quad -\infty < x < \infty, \quad (2.31)$$

and vice versa. The function  $\mathcal{M}$  is defined by (2.31) up to the right factor  $\exp(ixzD)a(z) \exp(-ixzD)$ . Normalization conditions

$$\lim_{x \rightarrow -\infty} \mathcal{M}(x, z) = I_m, \quad \overline{\lim}_{x \rightarrow \infty} \|\mathcal{M}(x, z)\| < \infty \quad (2.32)$$

are used to define  $\mathcal{M}$  uniquely. Sufficient conditions on the potential  $\zeta$ , under which relations (2.9) and (2.10) hold, follow from a particular case of the very useful Theorem 6.1 [6]:

**Theorem 2.5** *Suppose that the  $m \times m$  potential  $\zeta$  is two times differentiable, i.e.  $\zeta(x) \in C^2(-\infty, \infty)$ , and that  $\zeta^{(k)}(x) = \frac{d^k \zeta}{dx^k} \in L^1(-\infty, \infty)$  for  $k = 0, 1, 2$ . Then, for some  $M > 0$  the analytic in  $z$  matrix function  $\mathcal{M}(x, z)$ , which satisfies (2.31) and (2.32), is well defined in the domain  $\Im z < -M$ , the norm  $\|\mathcal{M}\|$  is uniformly bounded:*

$$\sup_{x \in (-\infty, \infty), \Im z < -M} \|\mathcal{M}(x, z)\| < \infty, \quad (2.33)$$

and uniformly with respect to  $x$  we have

$$\lim_{|z| \rightarrow \infty, \Im z < -M} \mathcal{M}(x, z) = I_m. \quad (2.34)$$

Finally, there is an  $m \times m$  matrix function  $\mathcal{M}_1(x) \in C^2(-\infty, \infty)$ , such that

$$\|\mathcal{M}(x, z) - I_m - \frac{\mathcal{M}_1(x)}{z}\| = O(z^{-2}) \quad (\Im z < -M), \quad (2.35)$$

and that  $\mathcal{M}_1^{(k)}(x) \in L^1(-\infty, \infty)$  for  $k = 1, 2$ .

Scheme of the proof. First, the case  $\|\zeta\|_1 < 1$  is treated in Theorem 3.8 [6]. The operator  $K_{z, \zeta}$  in the space of bounded matrix functions  $f(x)$  is introduced by the formula

$$K_{z, \zeta} f(x) = \int_x^\infty \exp(i(x-y)zD) (\zeta(y)f(y))_+ \exp(-i(x-y)zD) dy$$

$$- \int_{-\infty}^x \exp(i(x-y)zD) (\zeta(y)f(y))_- \exp(-i(x-y)zD) dy, \quad (2.36)$$

where  $g_+$  ( $g_-$ ) is the upper (lower) triangular part of the matrix  $g$ ,  $g = g_+ + g_-$ , and the main diagonal is included in  $g_-$ . One can see that  $\|K_{z,\zeta}\| \leq \|\zeta\|_1$  and so the operator  $I - K_{z,\zeta}$  is invertible. Moreover, it is proved that  $\mathcal{M}(x, z) = (I - K_{z,\zeta})^{-1}I_m$  satisfies (2.31) and (2.32), and all the properties of  $\mathcal{M}$  follow. The general case  $\|\zeta\|_1 < 2^r$  ( $r > 0$ ) follows by induction on  $r$  (Theorem A [6]). In Theorem 6.1 matrix functions  $\mathcal{M}_j(x)$  ( $j = 1, 2$ ) are introduced by the equalities  $\lim_{x \rightarrow -\infty} \mathcal{M}_j(x) = 0$ ,

$$\frac{d}{dx} \mathcal{M}_{j-1} + \zeta \mathcal{M}_{j-1} = i[D, \mathcal{M}_j^{od}], \quad \mathcal{M}_j^d(x) = - \int_{-\infty}^x (\zeta(y) \mathcal{M}_j^{od}(y))^d dy, \quad (2.37)$$

where  $\mathcal{M}_0 = I_m$ ,  $\mathcal{M}_j = \mathcal{M}_j^d + \mathcal{M}_j^{od}$ , and the entries of  $\mathcal{M}_j$  and  $\mathcal{M}_j^d$  coincide on the main diagonal, the entries of  $\mathcal{M}_j$  and  $\mathcal{M}_j^{od}$  coincide outside the main diagonal. Then, relation (2.35) follows from equality (6.14) in [6]:

$$\begin{aligned} \mathcal{M}(x, z)^{-1} \mathcal{M}^a(x, z) &= I_m - \int_x^\infty \exp(i(x-y)zD) g(y)_+ \exp(-i(x-y)zD) dy \\ &\quad + \int_{-\infty}^x \exp(i(x-y)zD) g(y)_- \exp(-i(x-y)zD) dy, \end{aligned} \quad (2.38)$$

where  $\mathcal{M}^a(x, z) = \sum_{j=0}^2 z^{-j} \mathcal{M}_j(x)$  and

$$g(x, z) = z^{-2} \mathcal{M}^{-1}(x, z) \left( \frac{d}{dx} \mathcal{M}_2(x) + \zeta(x) \mathcal{M}_2(x) \right).$$

### 3 Solvability of the inverse problem and Borg-Marchenko type result

Theorem 2.5 yields our next theorem.

**Theorem 3.1** *Suppose that the  $m \times m$  potential  $\zeta$  is two times differentiable, i.e.  $\zeta(x) \in C^2[0, \infty)$ , and that  $\zeta^{(k)}(x) = \frac{d^k \zeta}{dx^k} \in L^1[0, \infty)$  for  $k = 0, 1, 2$ . Then, for some  $M > 0$  a Weyl function of system (1.3) satisfies conditions (2.9) and (2.10).*

*Proof.* Define  $\zeta(x)$  on the semi-axis  $x < 0$  so that the conditions of Theorem 2.5 hold. Then we have

$$w(x, z) = \mathcal{M}(x, z) \exp(izx D) \mathcal{M}(0, z)^{-1} \quad (x \geq 0). \quad (3.1)$$

Hence, in view of Definition 2.1 and formulas (2.33) and (3.1), the function  $\varphi(z) = \mathcal{M}(0, z)$  is a Weyl function. Now, it is immediate from (2.35) that conditions (2.9) and (2.10) are fulfilled. ■

From the procedure to solve ISpP, a Borg-Marchenko type result follows.

**Theorem 3.2** *Let the analytic  $m \times m$  matrix functions  $\varphi_1$  and  $\varphi_2$  satisfy (2.9)-(2.11). Suppose that on some ray  $c\Im z = \Re z < 0$  ( $c \in \mathbb{R}$ ,  $\Im z < -M$ ) we have*

$$\varphi_1(z)^{-1} - \varphi_2(z)^{-1} = e^{-ilzD} O(z) \quad \text{for } |z| \rightarrow \infty. \quad (3.2)$$

*Then  $\varphi_1$  and  $\varphi_2$  are Weyl functions of systems (1.3) with potentials  $\zeta_1$  and  $\zeta_2$ , respectively, which satisfy (2.7) and the additional equality*

$$\zeta_1(x) \equiv \zeta_2(x) \quad (0 < x < l). \quad (3.3)$$

*Proof.* The fact that  $\varphi_1$  and  $\varphi_2$  are Weyl functions follows from Theorem 2.4. From (2.9)-(2.11) follow relations (2.12) and (2.13). According to the classical results on the Fourier transform in the complex domain (see, for instance, Theorem V in [28]), the function  $\frac{1}{z}\varphi(z)^{-1}$ , where  $\varphi(z)$  satisfies (2.12) and (2.13), admits Fourier representation. Moreover, taking into account also formula (2.14) and Plancherel's theorem, we get this representation, for  $z = \lambda - i\eta$  and fixed  $\eta > M$ , in terms of  $\Pi$ :

$$\frac{1}{z}\varphi(z)^{-1} = iD \int_0^\infty (\exp(-ixzD)) \Pi(x) dx, \quad (3.4)$$

where  $(\exp(-xMD))\Pi(x) \in L_{m \times m}^2(0, \infty)$ . As we have  $(\exp(-xMD))\Pi(x) \in L_{m \times m}^2(0, \infty)$ , so equalities (3.4) hold pointwise. Hence, we can use (3.4) to apply Phragmen-Lindelöf theorem. Namely, put

$$F(z) := (\exp(ilzD)) \int_0^l (\exp(-ixzD)) (\Pi_1(x) - \Pi_2(x)) dx, \quad (3.5)$$

where  $\Pi_1$  and  $\Pi_2$  correspond via formula (2.14) to  $\varphi_1$  and  $\varphi_2$ , respectively. By (3.4) and (3.5) we obtain

$$F(z) = \frac{-i}{z} D^{-1}(\exp(ilzD)) \left( \varphi_1(z)^{-1} - \varphi_2(z)^{-1} \right) - \int_l^\infty \left( \exp(i(l-x)zD) \right) \left( \Pi_1(x) - \Pi_2(x) \right) dx, \quad (3.6)$$

In view of (3.2), the relation

$$\left\| \frac{-i}{z} D^{-1}(\exp(ilzD)) \left( \varphi_1(z)^{-1} - \varphi_2(z)^{-1} \right) \right\| = O(1) \quad (3.7)$$

is true. Recall that  $z = \lambda - i\eta$ ,  $\eta > M$ , and that  $(\exp(-xMD))\Pi_k(x) \in L^2_{m \times m}(0, \infty)$ . It follows that

$$\left\| \int_l^\infty \left( \exp(i(l-x)zD) \right) \left( \Pi_1(x) - \Pi_2(x) \right) dx \right\| = O\left(\frac{1}{\sqrt{\eta}}\right), \quad \eta \rightarrow \infty. \quad (3.8)$$

According to (3.6)-(3.8), the matrix function  $F(z)$  is bounded on the ray  $c\Im z = \Re z$  ( $\Im z < -M$ ). It is immediate that  $F$  is bounded also on the axis  $\Im z = -M$ . Therefore, function  $F$  given by (3.5) satisfies conditions of the Phragmen-Lindelöf theorem in the angles with the boundaries  $\Im z = -M$  and  $c\Im z = \Re z$  ( $\Im z < -M$ ) in the lower semiplane. That is,  $F$  is bounded for  $\Im z \leq -M$ . It easily follows from (3.5) that  $F$  is bounded for  $\Im z > -M$  too, and that  $F(z) \rightarrow 0$  for  $z = \bar{z}$  tending to infinity. So we derive  $F \equiv 0$ . This implies that

$$\Pi_1(x) \equiv \Pi_2(x) \quad (0 < x < l). \quad (3.9)$$

Finally, notice that by (2.18), (2.19), and (2.25) the matrix function  $\Gamma(x, x)$  on the interval  $[0, l]$  is determined by  $\Pi(x)$  on the same interval. Hence, formulas (2.26) and (3.9) imply (3.3). ■

## 4 System with a shifted argument

In this section we shall consider system (1.3) with a shifted argument:

$$Y_x(x + \sigma, z) = (izD - \zeta(x + \sigma))Y(x + \sigma, z), \quad x \geq 0. \quad (4.1)$$

Taking into account normalization condition (2.2), for the fundamental solution  $w(x, \sigma, z)$  of (4.1) we have

$$w(x, \sigma, z) = w(x + \sigma, z)w(\sigma, z)^{-1}. \quad (4.2)$$

By (4.2) and Definition 2.1 the next proposition is immediate.

**Proposition 4.1** *Let  $\varphi(z)$  be a Weyl function of system (1.3). Then, the matrix function*

$$\varphi(\sigma, z) = w(\sigma, z)\varphi(z)\exp(-i\sigma zD) \quad (4.3)$$

*is a Weyl function of system (4.1).*

Notice that according to (2.24) the matrix function  $\Gamma(x, u)$  does not depend on the choice of  $l$  ( $l \geq x \geq u > 0$ ) for the domain of operators  $S$  and  $A$ . Putting in (2.24)  $x = u$ , we get

$$\Gamma(x, x) = s(x, x)D^{1/2} + \int_0^x s(x, v) V(x, v)^* dv. \quad (4.4)$$

By (2.19), (2.21), the first relation in (2.24) and equality (4.4), one can see that  $\Gamma(x, x)$  is continuous for  $x > 0$ . (In fact,  $\Gamma(x, x)$  is differentiable.) Therefore, according to (2.26) the matrix function  $\zeta(x)$  is continuous too. We shall put

$$\Gamma(0, 0) := \lim_{x \rightarrow +0} \Gamma(x, x), \quad (4.5)$$

and similar to (2.26) assume

$$\zeta(0) = (\Gamma(0, 0) - D\Gamma(0, 0)D^{-1})D^{1/2}. \quad (4.6)$$

Now, we can express  $\zeta(0)$  in terms of the matrix  $\alpha$ , which is defined by  $\varphi$  via representation (2.13). For that purpose introduce an  $m \times m$  matrix  $\hat{\alpha} = \{\hat{\alpha}_{kj}\}_{k,j=1}^m$  via the entries  $\alpha_{kj}$  of  $\alpha$ :

$$\hat{\alpha}_{kj} := \alpha_{kj} \quad \text{for } k \leq j; \quad \hat{\alpha}_{kj} := -\bar{\alpha}_{jk} \quad \text{for } k > j. \quad (4.7)$$

**Proposition 4.2** *Let the analytic  $m \times m$  matrix function  $\varphi$  satisfy (2.9)-(2.11). Then, for  $\zeta = \Omega(D, \varphi)$  we have*

$$\zeta(0) = i(D\hat{\alpha} - \hat{\alpha}D). \quad (4.8)$$

*Proof.* In view of (2.19) and (4.4) one obtains

$$\left(\Gamma(0, 0)\right)_{kj} = d_k^{-1} d_j^{\frac{1}{2}} \Pi'_{kj}(0) \text{ for } k \leq j; \quad \left(\Gamma(0, 0)\right)_{kj} = d_j^{-1} d_j^{\frac{1}{2}} \overline{\Pi}'_{jk}(0) \text{ for } k > j. \quad (4.9)$$

Recall that according to (2.16) we have  $\Pi'(0) = -iD\alpha$ , and so formulas (4.7) and (4.9) imply that  $\Gamma(0, 0) = -i\hat{\alpha}D^{\frac{1}{2}}$ . Hence, formula (4.8) follows from (4.6). ■

**Example 4.3** *Consider the simplest example  $m = 2$  and*

$$\zeta(x) \equiv \begin{bmatrix} 0 & -q \\ \bar{q} & 0 \end{bmatrix} = \text{const}, \quad (4.10)$$

*where const means a constant matrix. Calculate eigenvalues and eigenvectors of  $izD - \zeta$  to get*

$$izD - \zeta = T(z)\Lambda(z)T(z)^{-1}, \quad \Lambda(z) = \begin{bmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{bmatrix}, \quad (4.11)$$

$$T(z) = \begin{bmatrix} 1 & (izd_2 - \lambda_2(z))/\bar{q} \\ (\lambda_1(z) - izd_1)/q & 1 \end{bmatrix}, \quad (4.12)$$

*where  $\lambda_k$  are the roots of equation*

$$(izd_1 - \lambda)(izd_2 - \lambda) + |q|^2 = 0, \quad \text{i.e.} \quad (4.13)$$

$$\lambda_{1,2} = \frac{i}{2} \left( (d_1 + d_2)z \pm \sqrt{(d_1 - d_2)^2 z^2 + 4|q|^2} \right). \quad (4.14)$$

*In particular, we have*

$$\lambda_k(z) - izd_k = (-1)^{k+1} 2i|q|^2 \left( \sqrt{(d_1 - d_2)^2 z^2 + 4|q|^2} + (d_1 - d_2)z \right)^{-1}. \quad (4.15)$$

In view of (4.11) we get the fundamental solution

$$w(x, z) = T(z) \exp\{x\Lambda(z)\}T(z)^{-1}. \quad (4.16)$$

From (4.15) and (4.16) it follows that the matrix function  $\varphi(z) = T(z)$  satisfies (2.3), and so this  $\varphi$  proves a Weyl function of system (1.3) with  $\zeta$  of the form (4.10). Moreover, from (4.12) and (4.15) it follows that

$$\varphi(z) = T(z) = I_2 + \frac{i}{(d_1 - d_2)z} \begin{bmatrix} 0 & q \\ \bar{q} & 0 \end{bmatrix} + O\left(\frac{1}{z^3}\right). \quad (4.17)$$

Therefore conditions (2.9)-(2.11) are fulfilled.

**Remark 4.4** Notice that Example 4.3 describes the simplest case, where conditions (2.9) and (2.10) are fulfilled, but conditions of Theorem 3.1 are not valid.

Using Proposition 4.2 we obtain a similar result for a system with a shifted argument.

**Proposition 4.5** Let the analytic  $m \times m$  matrix function  $\varphi(z)$  satisfy (2.9)-(2.11), where

$$\alpha^* = -\alpha. \quad (4.18)$$

Then the matrix function  $\varphi(\sigma, z)$  also admits representation (2.9)-(2.11), where  $M(\sigma) = M + \varepsilon$  for an arbitrary fixed  $\varepsilon > 0$ , and where the matrix  $\alpha(\sigma)$  is such that

$$\alpha(\sigma)^* = -\alpha(\sigma). \quad (4.19)$$

Moreover, for  $\zeta = \Omega(D, \varphi)$  we have

$$\zeta(\sigma) = i(D\alpha(\sigma) - \alpha(\sigma)D). \quad (4.20)$$

**Remark 4.6** If the conditions of Theorem 3.1 are fulfilled, we have

$$\varphi(\sigma, z) = \mathcal{M}(\sigma, z), \quad \alpha(\sigma) = \mathcal{M}_1(\sigma). \quad (4.21)$$

According to the first relation in (2.37) and equality  $\mathcal{M}_0 \equiv I_m$ , we get

$$[D, \mathcal{M}_1(\sigma)] = -i\zeta(\sigma) = i\zeta(\sigma)^*, \quad \mathcal{M}_1^{od}(\sigma)^* = -\mathcal{M}_1^{od}(\sigma). \quad (4.22)$$

From the second equality in (2.37) and the first equality in (4.22) it follows that the  $k$ -th diagonal entry of  $\mathcal{M}_1$  has the form

$$\left(\mathcal{M}_1^d(\sigma)\right)_{kk} = -i \int_{-\infty}^{\sigma} \sum_{j \in \mathcal{N}_k} (d_j - d_k)^{-1} |\zeta_{kj}(y)|^2 dy = -\overline{\left(\mathcal{M}_1^d(\sigma)\right)_{kk}}, \quad (4.23)$$

where  $\mathcal{N}_k = \{j \in \mathbb{N} \mid 0 < j \leq m, j \neq k\}$ . Formulas (4.22) and (4.23) imply  $\mathcal{M}_1^* = -\mathcal{M}_1$ , and so, using (4.21), we derive  $\alpha^* = -\alpha$ ,  $\alpha(\sigma)^* = -\alpha(\sigma)$ . Thus, under conditions of Theorem 3.1, equality (4.18) is true and the statement of Proposition 4.5 follows from [6]. Still, the conditions of Proposition 4.5 are weaker than conditions of Theorem 3.1 (recall Example 4.3).

To prove Proposition 4.5 we shall need some preparations. In view of (2.18) we can present  $S^{-1}$  ( $S = S_l$ ) in the form

$$S^{-1}f = Tf = Df + \int_0^l T(x, u)f(u)du. \quad (4.24)$$

From  $ST = I$ , according to (2.18) and (4.24), it follows that

$$s(x, u)D + D^{-1}T(x, u) + \int_0^l s(x, v)T(v, u)dv = 0. \quad (4.25)$$

In particular, for the fixed values of  $u$  we shall assume

$$T(x, u) = -S^{-1}s(x, u)D, \quad (4.26)$$

and we shall define also  $T(x, u)$  pointwise by the formula

$$T(x, u) = -Ds(x, u)D + D \int_0^l \left(S^{-1}s(v, x)\right)^* s(v, u)dv D. \quad (4.27)$$

Introduce now an  $m \times m$  matrix function  $K(x)$  by the formula:

$$K(x) := D^{-1}\left(S^{-1}\Pi\right)(x) = \Pi(x) + D^{-1} \int_0^l T(x, u)\Pi(u)du. \quad (4.28)$$

From the identity (2.20), it follows that  $S^{-1}A - A^*S^{-1} = iS^{-1}\Pi\Pi^*S^{-1}$ , i.e.,

$$I_m + D^{-1} \int_u^l T(x, v) dv + \int_x^l T(v, u) dv D^{-1} = K(x)K(u)^*. \quad (4.29)$$

Using (4.27) and (4.28), it is shown in the Appendix 1 that  $K(x)$  is continuous and differentiable. The following lemma is also proved in the Appendix 1.

**Lemma 4.7** *Let the analytic  $m \times m$  matrix function  $\varphi$  satisfy (2.9)-(2.11), and (4.18). Then, the relations*

$$z \left( w(l, z) \varphi(z) \exp(-ilzD) - I_m - \frac{i}{z} K(l) \left( K'(l) \right)^* D^{-1} \right) \in L_{m \times m}^2(-\infty, \infty), \quad (4.30)$$

and

$$\sup \left\| z \left( w(l, z) \varphi(z) \exp(-ilzD) - I_m \right) \right\| < \infty \quad (4.31)$$

are true for  $\Im z < -M - \varepsilon$  (for any  $\varepsilon > 0$ ).

**Proof of Proposition 4.5.** By formula (4.3) and Lemma 4.7 we see that  $\varphi(l, z)$  satisfies (2.9)-(2.11), and

$$\alpha(l) = iK(l) \left( K'(l) \right)^* D^{-1}. \quad (4.32)$$

Taking into account (4.27) we get, that  $T(l, u)$  is continuous in  $u$  at  $u = l$ . Thus, putting  $x = l$  and differentiating both sides of (4.29) with respect to  $u$  at  $u = l$ , we derive

$$K(l) \left( K'(l) \right)^* = -D^{-1} T(l, l) D^{-1}. \quad (4.33)$$

Hence, according to (4.32) and (4.33), we have

$$\alpha(l) = -iD^{-1} T(l, l) D^{-1}. \quad (4.34)$$

From (2.16), (2.19), and (4.18) it follows that  $s(l, l) = s(l, l)^*$ . Therefore, formula (4.27) implies  $T(l, l) = T(l, l)^*$ , and so, in view of (4.34), we have (4.19). According to (4.19), the equality  $\widehat{\alpha}(\sigma) = \alpha(\sigma)$  is true, where

$$\widehat{\alpha}_{kj}(\sigma) := \alpha_{kj}(\sigma) \quad \text{for } k \leq j; \quad \widehat{\alpha}_{kj}(\sigma) := -\overline{\alpha}_{jk}(\sigma) \quad \text{for } k > j.$$

Now, as  $\varphi(\sigma, z)$  satisfies (2.9)-(2.11), formula (4.20) follows from Proposition 4.2. ■

**Remark 4.8** Notice that under conditions of Theorem 2.4 formulas (2.14), (2.18), (2.19), (2.25), and (2.26) define a solution  $\zeta$  of the inverse problem even without the requirement  $d_k > d_j$  for  $k > j$ . That is, if  $D > 0$  and  $d_k \neq d_j$  for  $k \neq j$ , then  $\zeta$ , which is recovered from  $\varphi$  by the mentioned above formulas, satisfies (2.7) and defines such a system that (2.3) holds. By Theorem 2.3 the solution of the inverse problem is unique. Recall that we denote this solution  $\zeta$  by  $\Omega(D, \varphi)$ .

Consider now nonlinear optics ( $N$ -wave) equation (1.1), where  $u = u^*$  and  $D$  satisfies the second relation in (1.4).

**Remark 4.9** We assume for convenience that the entries of  $u$  on the main diagonal are identical zeros, i.e.,  $u_{kk} \equiv 0$ .

Suppose first that the Weyl function  $\varphi$  is bounded and satisfies (2.8). Then [31], for the case

$$\check{D} = \text{diag} \{ \check{d}_1, \check{d}_2, \dots, \check{d}_m \}, \quad \check{d}_1 > \check{d}_2 > \dots > \check{d}_m > 0, \quad (4.35)$$

the initial condition

$$[D, u(x, 0)] = \Omega(D, \varphi) \quad (4.36)$$

defines at most one continuously differentiable solution  $u$  of (1.1) on the semi-band  $x \geq 0, \quad \varphi \geq t \geq 0$ .

Consider now the more general case, where  $D$  satisfies the second relation in (1.4),  $\check{D} > 0$  and  $\check{d}_k \neq \check{d}_j$  for  $k \neq j$ , but the inequalities in (4.35) do not necessarily hold. Define the initial condition via Weyl function  $\varphi$  by formula (4.36) and define the boundary condition via the same  $\varphi$ :

$$[\check{D}, u(0, t)] = \Omega(\check{D}, \varphi). \quad (4.37)$$

Assume that the analytic  $m \times m$  matrix function  $\varphi$  satisfies (2.9)-(2.11), and (4.18). Then, we have [31]:

a) The evolution of the Weyl function is given by the formula

$$\varphi(t, z) = R(t, z) \varphi(z) \exp(-iz\check{D}t), \quad (4.38)$$

where

$$\frac{dR(t, z)}{dz} = \left( iz\check{D} - \Omega(\check{D}, \varphi) \right) R(t, z), \quad R(0, z) = I_m. \quad (4.39)$$

Moreover, the matrix functions  $\varphi(t, z)$  also satisfy conditions (2.9)-(2.11), and (4.18) for some matrices  $\alpha(t)$ .

b) The matrix function  $u(x, t) = u(x, t)^*$  is well-defined by the relation

$$[D, u(x, t)] = \Omega(D, \varphi(t, z)) \quad (4.40)$$

and satisfies nonlinear optics equation (1.1) and initial-boundary value conditions (4.36) and (4.37).

## 5 System with a shifted argument and Darboux matrices

From formula (1.3) it follows that  $\varphi(\sigma, z)$  given by (4.3) satisfies equation

$$\frac{d\varphi(\sigma, z)}{d\sigma} = iz \left( D\varphi(\sigma, z) - \varphi(\sigma, z)D \right) - \zeta(\sigma)\varphi(\sigma, z). \quad (5.1)$$

The next proposition is also true.

**Proposition 5.1** *Let  $\varphi(\sigma, z)$  satisfy equation (5.1), and let  $\varphi(0, z)$  be a Weyl function of system (1.3). Then, the matrix functions  $\varphi(\sigma, z)$  with fixed  $\sigma > 0$  are Weyl functions of systems (4.1).*

*Proof.* By (5.1) one can see that  $\varphi(\sigma, z)e^{iz\sigma D}$  satisfies (4.1), i.e.,

$$\varphi(\sigma, z)e^{iz\sigma D}\varphi(0, z)^{-1} = w(\sigma, z).$$

Hence,  $\varphi(\sigma, z)$  has the form (4.3), and our proposition follows from Proposition 4.1. ■

Notice that equation (5.1) coincides with the definition of the Darboux matrix, which transforms solution  $e^{izx D}$  of the auxiliary to nonlinear optics equation system with a trivial potential  $\zeta_0 = 0$  into solution of system (1.3). Therefore, by constructing Darboux matrices we obtain examples of the Weyl functions. We propose below two schemes to construct Darboux matrices. The first scheme is a particular case of the so called GBDT (see [32, 34, 37] and references therein). Namely, we shall introduce Darboux matrix as a

transfer matrix function (in Lev Sakhnovich form) with additional dependence on the variable  $x$ :

$$w_A(x, z) = I_m - i\Pi(x)^* S(x)^{-1} (A - zI_m)^{-1} \Pi(x). \quad (5.2)$$

Distinct from formula (2.28), both  $\Pi(x)$  and  $S(x)$  in the GBDT method are differentiable matrix functions, where  $\Pi$  is determined by the linear differential system. Correspondingly, the factor  $\Pi(x)^*$  above means multiplication by the matrix adjoint to  $\Pi(x)$ . The second scheme is also constructed in the spirit of the GBDT approach.

**Scheme 1.** This scheme is precisely GBDT for system (1.3) (see [30, 32]). First, we fix an integer  $n > 0$ , two  $n \times n$  parameter matrices  $A$  and  $S(0)$ , and  $n \times m$  matrix  $\Pi(0)$  such that

$$AS(0) - S(0)A^* = i\Pi(0)\Pi(0)^*. \quad (5.3)$$

Introduce  $\Pi(x)$  and  $S(x)$  for  $x > 0$  by the equations

$$\Pi_x = -iA\Pi D + \Pi\zeta, \quad S_x = \Pi D \Pi^*, \quad (5.4)$$

and put

$$\tilde{\zeta}(x) = \zeta(x) - (D\Pi(x)^* S(x)^{-1} \Pi(x) - \Pi(x)^* S(x)^{-1} \Pi(x) D). \quad (5.5)$$

Then, in the points of invertibility of  $S(x)$ , the transfer matrix function  $w_A$  satisfies [32] the equation

$$\frac{d}{dx} w_A(x, z) = \tilde{G}(x, z) w_A(x, z) - w_A(x, z) G(x, z), \quad (5.6)$$

where

$$\tilde{G}(x, z) = izD - \tilde{\zeta}(x), \quad G(x, z) = izD - \zeta(x). \quad (5.7)$$

By (5.3) and (5.4) we have also

$$AS(x) - S(x)A^* = i\Pi(x)\Pi(x)^*. \quad (5.8)$$

Further we assume that  $S(0) > 0$ . Then, according to (5.4), we have  $S(x) > 0$ , and so  $S(x)$  is invertible. To make the paper self-sufficient we give the proof of formula (5.6) in Appendix 2.

**Scheme 2.** Fix an interval  $[a, b]$  and an  $m \times m$  weight matrix function  $\rho(t) > 0$ , which is bounded on this interval. Thus, the space  $L_m^2(\rho)$  with the scalar product  $(f, g) = \int_a^b g(t)^* \rho(t) f(t) dt$  is generated. Let  $\zeta(x)$  be continuous, and define operators  $A$  and  $S(x)$  in  $L_m^2(\rho)$  by the formulas

$$Af = tf(t), \quad S(x)f = cf + i \int_a^b \frac{w(x, t)^* w(x, y)}{t - y} \rho(y) f(y) dy, \quad (5.9)$$

respectively. Here we take the principal value of the integral in (5.9). Notice that  $\frac{d}{dx} w(x, t)^* w(x, y) = -i(t - y)w(x, t)^* Dw(x, y)$ , i.e.,

$$(t - y)^{-1} w(x, t)^* w(x, y) = (t - y)^{-1} - i \int_0^x w(u, t)^* Dw(u, y) du. \quad (5.10)$$

Hence, we can rewrite the expression for  $S$  from (5.9) in the form

$$S(x)f = cf + i \int_a^b (t - y)^{-1} \rho(y) f(y) dy + \int_a^b \int_0^x w(u, t)^* Dw(u, y) du \rho(y) f(y) dy. \quad (5.11)$$

As  $\rho$  is bounded and the operator  $\int_a^b (t - y)^{-1} \bullet dy$  in  $L_m^2$  is bounded, so the operator  $\int_a^b (t - y)^{-1} \rho(y) \bullet dy$ , and also operator  $S(x)$ , is bounded in  $L_m^2(\rho)$ . Next, introduce operator  $\Pi(x)$ , acting from  $\mathbb{C}^m$  into  $L_m^2(\rho)$ , and operator  $\Pi(x)^*$ :

$$\Pi(x)f = w(x, t)^* f, \quad \Pi(x)^* g = \int_a^b w(x, y) \rho(y) g(y) dy. \quad (5.12)$$

By (5.11) and (5.12) we have the second equality in (5.4), that is,  $S_x = \Pi D \Pi^* \geq 0$ . Now, choose  $c$  so that

$$cI + i \int_a^b (t - y)^{-1} \rho(y) \bullet dy > 0. \quad (5.13)$$

As  $S_x \geq 0$ , by (5.13) we have  $S(x) > 0$ . In view of (1.3), definition (5.9) of  $A$  and definition (5.12), the first equality in (5.4) is true too. Moreover, according to (5.9) the identity (5.8) holds. By (5.4) and (5.8) the equation (5.6) is satisfied, where  $\tilde{G}$  and  $G$  are given via (5.5) and (5.7), - see Appendix B. Using Definition 2.1, Proposition 4.1 and formula (5.6) one easily gets the next proposition.

**Proposition 5.2** *Let the Darboux matrix  $w_A$  be defined via (5.2), using Scheme 1 or Scheme 2. Then the normalized fundamental solution of the transformed system*

$$\tilde{Y}_x(x, z) = \tilde{G}(x, z)\tilde{Y}(x, z) = (izD - \tilde{\zeta}(x))\tilde{Y}(x, z) \quad (5.14)$$

*is given by the formula*

$$\tilde{w}(x, z) = w_A(x, z)w(x, z)w_A(0, z)^{-1}, \quad (5.15)$$

*where  $w$  is the fundamental solution of the initial system (1.3). Suppose also that  $\varphi(z)$  is a Weyl function of system (1.3). Then the matrix function  $w_A(0, z)\varphi(z)$  is a Weyl function of the system (5.14) and*

$$\tilde{\varphi}(\sigma, z) = w_A(\sigma, z)w(\sigma, z)\varphi(z)\exp(-i\sigma zD) \quad (5.16)$$

*is a Weyl function of the system  $\tilde{Y}_x(x + \sigma, z) = \tilde{G}(x + \sigma, z)\tilde{Y}(x + \sigma, z)$  with a shifted argument.*

Next we shall consider 2 simple examples of Scheme 1, including the case of  $A$  non-diagonal.

**Example 5.3** *Let*

$$\zeta(x) \equiv 0, \quad n = 2, \quad \Pi(0) = [f_1 \quad f_2 \quad \dots \quad f_m] \quad (f_k \in \mathbb{C}^2). \quad (5.17)$$

*It follows from the first relation in (5.4) that*

$$\Pi(x) = \begin{bmatrix} \exp(-ixd_1A)f_1 & \exp(-ixd_2A)f_2 & \dots & \exp(-ixd_mA)f_m \end{bmatrix}. \quad (5.18)$$

*It is immediate that*

$$\exp(-ixdA) = \text{diag}\{\exp(-ixda), \exp(-ixda)\} \quad \text{for } A = \text{diag}\{a, a\}, \quad (5.19)$$

*and*

$$\exp(-ixdA) = \exp(-ixda) \left( I_2 - ixd \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \quad \text{for } A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}. \quad (5.20)$$

Finally, assuming  $a \neq \bar{a}$ , formula (5.8) implies

$$S = \{s_{kj}\}_{k,j=1}^2 = i(a - \bar{a})^{-1} \Pi \Pi^*$$

for the case (5.19). For the case (5.20) the same formula implies

$$\begin{aligned} s_{22} &= i(a - \bar{a})^{-1} (\Pi \Pi^*)_{22}, \quad s_{12} = \overline{s_{21}} = (a - \bar{a})^{-1} \left( i(\Pi \Pi^*)_{12} - s_{22} \right), \\ s_{11} &= (a - \bar{a})^{-1} \left( i(\Pi \Pi^*)_{11} + s_{12} - s_{21} \right). \end{aligned}$$

Substitute explicit formulas for  $\Pi$  and  $S$ , which are given above, into (5.2) and (5.5) to obtain explicit formulas for  $w_A$  and  $\tilde{\zeta}$ .

Notice that  $w_A(\sigma, z)$  admits representation

$$w_A(\sigma, z) = I_m + \frac{i}{z} \Pi(\sigma)^* S(\sigma)^{-1} \Pi(\sigma) + O\left(\frac{1}{z^2}\right), \quad z \rightarrow \infty. \quad (5.21)$$

From Proposition 5.2 and formula (5.21) follows corollary.

**Corollary 5.4** *Let the conditions of Proposition 5.2 be fulfilled, and let a Weyl function  $\varphi$  of the initial system (1.3) satisfy (2.9)-(2.11) with the corresponding matrix  $\alpha$ . Then, for some  $M > 0$ , a Weyl function  $\tilde{\varphi}(0, z)$  of the transformed system (5.14) satisfies formulas (2.9)-(2.11), where matrix  $\alpha$  is substituted by the matrix  $\tilde{\alpha}$ :*

$$\tilde{\alpha} = \alpha + i\Pi(0)^* S(0)^{-1} \Pi(0). \quad (5.22)$$

From Propositions 4.5 and 5.2 and from formula (5.21) we get.

**Corollary 5.5** *Let the conditions of Proposition 5.2 be fulfilled, and let a Weyl function  $\varphi$  of the initial system (1.3) satisfy formulas (2.9)-(2.11) and (4.18) with the corresponding matrix  $\alpha$ . Then, for some  $M > 0$ , a Weyl function  $\tilde{\varphi}(\sigma, z)$  of the transformed system with a shifted by  $\sigma$  argument satisfies formulas (2.9)-(2.11), (4.18), where matrix  $\alpha(\sigma)$  is substituted by the matrix  $\tilde{\alpha}(\sigma)$ :*

$$\tilde{\alpha}(\sigma) = \alpha(\sigma) + i\Pi(\sigma)^* S(\sigma)^{-1} \Pi(\sigma), \quad \tilde{\zeta}(\sigma) = i(D\tilde{\alpha}(\sigma) - \tilde{\alpha}(\sigma)D). \quad (5.23)$$

Formulas (5.23) yield (5.5).

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## A Appendix

In this Appendix we shall obtain some properties of the matrix function  $K(x)$  defined in (4.28), and using these properties we shall prove Lemma 4.7.

**Lemma A.1** *Let the analytic  $m \times m$  matrix function  $\varphi$  satisfy (2.9)-(2.11), and (4.18). Then,  $K(x)$  is twice differentiable and satisfies equalities:*

$$I_m - (\Pi'(u), K(u)) = K(0)^*, \quad (\text{A.1})$$

$$(D^{-1}\Pi''(u), K(u)) - (K'(0))^* D^{-1} + \left( I_m - (\Pi'(u), K(u)) \right) D^{-1} \Pi'(0) = 0. \quad (\text{A.2})$$

**Proof.** According to (2.9)-(2.11) and (2.14), the matrix function  $\Pi(x)$  is two times differentiable. By (4.27) the matrix function  $T(x, u) + Ds(x, u)D$  is continuous in both variables. Then, in view of formula (4.28)  $K(x)$  is continuous. Consider now formula (4.29). It follows that  $K(l)K(l)^* = I_m$ . Hence, for the case  $u = l$  formula (4.29) yields

$$K(x) = \left( I_m + \int_x^l T(v, u) dv D^{-1} \right) K(l). \quad (\text{A.3})$$

It follows from (A.3) that  $K(x)$  is differentiable and

$$K'(x) = -T(x, l) D^{-1} K(l). \quad (\text{A.4})$$

Put now in (4.29)  $x = l$ , multiply both sides from the right by  $\Pi'(u)$ , and integrate the obtained expressions with respect to  $u$  from 0 to  $l$ . We have

$$\Pi(l) - \Pi(0) + D^{-1} \int_0^l \int_u^l T(l, v) dv \Pi'(u) du = K(l) \int_0^l K(u)^* \Pi'(u) du. \quad (\text{A.5})$$

Recall that by (2.16) we have  $\Pi(0) = I_m$  and change also the order of integration in (A.5). Then we derive

$$\Pi(l) - I_m + D^{-1} \int_0^l T(l, v)(\Pi(v) - I_m)dv = K(l) \int_0^l K(u)^* \Pi'(u)du. \quad (\text{A.6})$$

Using (4.28), rewrite (A.6) in the form

$$K(l) - I_m - D^{-1} \int_0^l T(l, v)dv = K(l) \int_0^l K(u)^* \Pi'(u)du. \quad (\text{A.7})$$

According to (4.29) we get

$$I_m + D^{-1} \int_0^l T(l, v)dv = K(l)K(0)^*. \quad (\text{A.8})$$

Recalling that  $K(l)K(l)^* = I_m$  and using (A.7) and (A.8), we finally obtain (A.1). By (4.18) and the second relation in (2.16) we have

$$D^{-1}\Pi'(0) = \left(D^{-1}\Pi'(0)\right)^*. \quad (\text{A.9})$$

Hence, according to (2.19),  $s(x, u)$  is continuous, and so  $T(x, u)$  is continuous. Moreover, in view of (4.27) and (A.4) one can see that  $K$  is two times differentiable and the entries of  $K''(x)$  belong  $L^2(0, l)$ . Formula (2.19) yields also

$$D^{-1}\Pi'(x) = s(x, 0). \quad (\text{A.10})$$

Put again in (4.29)  $x = l$ , multiply both sides from the right by  $D^{-1}\Pi''(u)$ , and integrate the obtained expressions with respect to  $u$  from 0 to  $l$ :

$$\begin{aligned} D^{-1}(\Pi'(l) - \Pi'(0)) + D^{-1} \int_0^l T(l, v)D^{-1}(\Pi'(v) - \Pi'(0))dv \\ = K(l) \left(D^{-1}\Pi''(u), K(u)\right). \end{aligned}$$

So, taking into account (A.10), we have

$$D^{-1} \left( Ds(l, 0) + \int_0^l T(l, v)s(v, 0)dv \right) - \left( I_m + \int_0^l T(l, v)dv \right) D^{-1}\Pi'(0)$$

$$= K(l) \left( D^{-1} \Pi''(u), K(u) \right). \quad (\text{A.11})$$

By (4.24), (4.26), and (4.29), we rewrite (A.11) as

$$-D^{-1}T(l, 0)D^{-1} - K(l)K^*(0)D^{-1}\Pi'(0) = K(l) \left( D^{-1}\Pi''(u), K(u) \right). \quad (\text{A.12})$$

Recall that  $S^{-1} = (S^{-1})^*$  and that  $T(x, u)$  is continuous. Hence, it follows that  $T(x, u) = T(u, x)^*$  and, in particular, that  $T(l, 0) = T(0, l)^*$ . Therefore, using (A.4) and the equality  $K(l)K(l)^* = I_m$ , we obtain

$$T(l, 0) = -DK(l) \left( K'(0) \right)^*. \quad (\text{A.13})$$

From (A.12) and (A.13) we get

$$\left( K'(0) \right)^* D^{-1} - K^*(0)D^{-1}\Pi'(0) = \left( D^{-1}\Pi''(u), K(u) \right). \quad (\text{A.14})$$

Finally, (A.1) and (A.14) imply (A.2). ■

**Proof of Lemma 4.7.** For the proof of lemma we shall use representations (2.27) and (2.28). First consider expression  $(I - zA)^{-1}\Pi$  from (2.28). It is easy to see that

$$\left( (I - zA)^{-1}\Pi \right)(x) = \Pi(x) + izD \int_0^x \exp(i(x-u)zD) \Pi(u) du. \quad (\text{A.15})$$

From (3.4) and (A.15), using integration by parts, we obtain

$$\begin{aligned} & \left( (I - zA)^{-1}\Pi \right)(x) \\ &= \Pi(x) + (\exp izx zD) \left( \varphi(z)^{-1} - izD \int_x^\infty \exp(-iuz zD) \Pi(u) du \right) \\ &= (\exp izx zD) \varphi(z)^{-1} + \frac{i}{z} D^{-1} \left( \Pi'(x) + \int_x^\infty \exp(i(x-u)zD) \Pi''(u) du \right). \end{aligned} \quad (\text{A.16})$$

By (2.28) and (A.16) we have

$$\begin{aligned} w_A(l, z) \varphi(z) \exp(-ilzD) &= \varphi(z) \exp(-ilzD) + iz \Pi^* S^{-1} \left( \exp(i(x-l)zD) \right. \\ &\quad \left. + \frac{i}{z} D^{-1} \left( \Pi'(x) + \int_x^\infty \exp(i(x-u)zD) \Pi''(u) du \right) \varphi(z) \exp(-ilzD) \right). \end{aligned} \quad (\text{A.17})$$

To consider the asymptotics of the right-hand side of (A.17) we take into account that  $\Pi^* S^{-1} D^{-1}$  acts as the operator  $\int_0^l K(x)^* \bullet dx$ . Then, using integration by parts, we get

$$\begin{aligned} iz\Pi^* S^{-1} \exp(i(x-l)zD) &= K(l)^* - K(0)^* \exp(-ilzD) \\ &\quad - \int_0^l K'(x)^* \exp(ixzD) dx \exp(-ilzD). \end{aligned} \quad (\text{A.18})$$

Use integration by parts again to rewrite (A.18) in the form

$$\begin{aligned} iz\Pi^* S^{-1} \exp(i(x-l)zD) &= K(l)^* + \frac{i}{z} K'(l)^* D^{-1} - K(0)^* \exp(-ilzD) \\ &\quad - \frac{i}{z} \left( K'(0)^* D^{-1} \exp(-ilzD) + q(l, z) \right), \end{aligned} \quad (\text{A.19})$$

where

$$q(l, z) := \int_0^l K''(x)^* D^{-1} \exp(ixzD) dx \exp(-ilzD),$$

and so, for any  $\varepsilon > 0$  we have

$$\sup_{\Im z < -M-\varepsilon} \|q(l, z)\| < \infty; \quad (\text{A.20})$$

and for the lines with the fixed values of  $\Im z$  we have

$$q(l, z) \in L_{m \times m}^2(-\infty, \infty) \quad (\Im z < -M - \varepsilon, \quad -\infty < \Re z < \infty). \quad (\text{A.21})$$

Consider now two other terms on the right-hand side of (A.17) and take into account the second relation in (2.16) and (A.1) as well as the asymptotics of  $\varphi$ , that is, formulas (2.9) and (2.10) to obtain

$$\begin{aligned} &\varphi(z) \exp(-ilzD) - \Pi^* S^{-1} D^{-1} \Pi'(x) \varphi(z) \exp(-ilzD) \\ &= K(0)^* \left( I_m + \frac{i}{z} D^{-1} \Pi'(0) \right) \exp(-ilzD) + \frac{1}{z} q_1(l, z), \end{aligned} \quad (\text{A.22})$$

where  $q_1$  satisfies (A.20) and (A.21). Finally, from integration by parts and asymptotics of  $\varphi$  it follows that

$$-\Pi^* S^{-1} D^{-1} \int_x^\infty \exp(i(x-u)zD) \Pi''(u) du \varphi(z) \exp(-ilzD)$$

$$= \frac{i}{z} \left( \int_0^l K(x)^* D^{-1} \Pi''(x) dx + q_2(l, z) \right) = \frac{i}{z} \left( \left( D^{-1} \Pi''(u), K(u) \right) + q_2(l, z) \right), \quad (\text{A.23})$$

where  $q_2$  satisfies (A.20) and (A.21). In view of (A.1) and (A.2) the sum of the right-hand sides of (A.19), (A.22) and (A.23) equals  $K(l)^* + (i/z)K'(l)^*D^{-1} + q_3(l, z)/z$ , where  $q_3$  satisfies (A.20) and (A.21). In other words we have

$$w_A(l, z)\varphi(z) \exp(-ilzD) = K(l)^* + \frac{i}{z}K'(l)^*D^{-1} + \frac{1}{z}q_3(l, z). \quad (\text{A.24})$$

Further notice that in view of (4.26) and equality  $T(x, u) = T(u, x)^*$  we have

$$(S^{-1}s(u, l))^* = (-T(u, l)D^{-1})^* = -D^{-1}T(l, u). \quad (\text{A.25})$$

According to (2.29), (4.24) and (A.25) it follows that

$$\beta(l) = D^{\frac{1}{2}}D^{-1}(S^{-1}\Pi)(l). \quad (\text{A.26})$$

Compare (4.28) and (A.26) to get

$$\beta(l) = D^{\frac{1}{2}}K(l). \quad (\text{A.27})$$

By (2.27), (A.24), and (A.26) we obtain (4.30) and (4.31). ■

## B Appendix

**Proof** of formula (5.6).

From (5.4) and (5.8) it follows that

$$\left( \Pi^* S^{-1} \right)_x = iD\Pi^* A^* S^{-1} - \zeta \Pi^* S^{-1} - \Pi^* S^{-1} \Pi D \Pi^* S^{-1}. \quad (\text{B.1})$$

By (5.8) we have  $A^* S^{-1} = S^{-1}A - iS^{-1}\Pi\Pi^* S^{-1}$ , and so formula (B.1) can be rewritten as

$$\begin{aligned} \left( \Pi^* S^{-1} \right)_x &= (D\Pi^* S^{-1}\Pi - \Pi^* S^{-1}\Pi D - \zeta)\Pi^* S^{-1} + iD\Pi^* S^{-1}A \\ &= -\tilde{\zeta}\Pi^* S^{-1} + iD\Pi^* S^{-1}A, \end{aligned} \quad (\text{B.2})$$

where  $\tilde{\zeta}$  is defined in (5.5). Now, from the definition (5.2) of  $w_A$  and formulas (5.4) and (B.2) it follows that

$$\begin{aligned} \frac{dw_A}{dx} = & -\tilde{\zeta}(w_A - I_m) + iD\Pi^*S^{-1}A(A - zI_m)^{-1}\Pi - (w_A - I_m)(-\zeta) \\ & - \Pi^*S^{-1}(A - zI_m)^{-1}A\Pi D. \end{aligned} \quad (\text{B.3})$$

Substitute  $A = (A - zI_m) + zI_m$  into the second and fourth terms on the right-hand side to rewrite (B.3) as

$$\frac{dw_A}{dx} = (izD - \tilde{\zeta})(w_A - I_m) - (w_A - I_m)(izD - \zeta) + D\Pi^*S^{-1}\Pi - \Pi^*S^{-1}\Pi D. \quad (\text{B.4})$$

Formulas (5.5) and (B.4) imply

$$\frac{dw_A}{dx} = (izD - \tilde{\zeta})(w_A - I_m) - (w_A - I_m)(izD - \zeta) + \zeta - \tilde{\zeta}, \quad (\text{B.5})$$

and (5.6) is immediate.

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